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# Non-antisymmetric versions of Nambu–Poisson and algebroid brackets

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## Abstract

We show that we can skip the skew-symmetry assumption in the definition of Nambu–Poisson brackets. In other words, an  $n$ -ary bracket on the algebra of smooth functions which satisfies the Leibniz rule and an  $n$ -ary version of the Jacobi identity must be skew symmetric. A similar result holds for a non-antisymmetric version of Lie algebroids.

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## 1. Introduction

Two main directions have been suggested for the generalization of the notion of a Lie algebra. First, Filippov developed a proposal for brackets with more than two arguments, i.e.  $n$ -ary brackets. In [Fi] he proposed a definition of such structures (which we shall call *Filippov algebras*) with a version of the Jacobi identity for  $n$ -arguments (we shall call this the *Filippov identity*, FI):

$$\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = \sum_{k=1}^n \{g_1, \dots, \{f_1, \dots, f_{n-1}, g_k\}, \dots, g_n\}. \quad (1)$$

Note that in the binary case ( $n = 2$ ), the FI coincides with the Jacobi identity. Independently, Nambu [Na], looking for generalized formulations of Hamiltonian mechanics, found  $n$ -ary analogues of Poisson brackets and then Takhtajan [Ta], motivated by physics, rediscovered the FI (and called it the fundamental identity) for these. The *Filippov brackets* are assumed to be  $n$ -linear and skew symmetric and *Nambu–Poisson brackets*, defined on algebras of smooth functions, satisfy additionally the *Leibniz rule*:

$$\{f_1 f'_1, f_2, \dots, f_n\} = f_1 \{f'_1, f_2, \dots, f_n\} + \{f_1, f_2, \dots, f_n\} f'_1. \quad (2)$$

On the other hand, Loday (cf [Lo]), while studying relations between Hochschild and cyclic homology in the context of searching for obstructions to the periodicity of algebraic K-theory, discovered that one can skip the skew-symmetry assumption in the definition of a Lie algebra, still having a possibility of defining appropriate (co-) homology (see [Lo, LP] and [Lo1], chapter 10.6). His Jacobi identity for such structures was formally the same as the classical Jacobi identity in the form of (1) for  $n = 2$ :

$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}. \quad (3)$$

This time, however, this is no longer equivalent to

$$\{\{f, g\}, h\} = \{\{f, h\}, g\} + \{f, \{g, h\}\} \quad (4)$$

since we have no skew symmetry. Loday called such structures *Leibniz algebras* but, since we have already associated the name of Leibniz with the Leibniz identity, we shall call them *Loday algebras*. This is in accordance with the terminology of [KS], where analogous structures in the graded case are defined. Of course, there is no particular reason not to define Loday algebras by means of (4) instead of (3) (and, in fact, this was the original Loday definition), but both categories are equivalent via transposition of arguments. Similarly, for associative algebras we can obtain associated algebras by transposing arguments, but in this case we still obtain associative algebras. It is interesting that Nambu–Poisson brackets lead to some Loday algebras and hence to the corresponding (co-) homology (see [DT]).

It is now clear that we can combine the two generalizations and define *Filippov–Loday algebras* as those which are equipped with  $n$ -ary brackets, not skew symmetric in general, but satisfying the FI. We can also define a Loday version of Nambu–Poisson algebras or rings (we shall call them *Nambu–Poisson–Loday*, or simply *Nambu–Loday*, algebras (or rings)), assuming additionally that a Filippov–Loday structure is defined on a commutative associative algebra (or ring respectively) and satisfies the Leibniz rule (with respect to all arguments separately, since we have no skew symmetry).

In this short paper we first deal with the problem of finding examples of new, i.e. non antisymmetric, Nambu–Poisson–Loday brackets. The result is, to some extent, unexpected. We show that for a wide variety of associative commutative algebras, including algebras of smooth functions, we obtain nothing more than what we already know, since Nambu–Loday algebras have to be skew symmetric. In particular, we can skip the skew-symmetry axiom in the standard definition of the Poisson bracket.

We obtain a similar negative result for a Loday-type generalization of Lie algebroids: they are locally, in principle, skew symmetric, or they are bundles of Loday algebras.

## 2. Main theorem

**Definition.** Let  $A$  be an associative commutative algebra. Let  $\{\cdot, \dots, \cdot\}$  be an  $n$ -ary bracket on  $A$ , i.e. an operation with  $n$ -arguments

$$A \times \dots \times A \ni (f_1, \dots, f_n) \mapsto \{f_1, \dots, f_n\} \in A \quad (5)$$

which is linear with respect to all arguments:

$$\{f_1, \dots, \alpha f_i + \beta f'_i, \dots, f_n\} = \alpha \{f_1, \dots, f_i, \dots, f_n\} + \beta \{f_1, \dots, f'_i, \dots, f_n\}. \quad (6)$$

We shall call such a bracket a *Nambu–Loday bracket*, if it satisfies the following two conditions:

(i) the Leibniz rule (with respect to each argument),

$$\{f_1, \dots, f_i f'_i, \dots, f_n\} = f_i \{f_1, \dots, f'_i, \dots, f_n\} + \{f_1, \dots, f_i, \dots, f_n\} f'_i \quad (7)$$

for all  $i = 1, \dots, n$ , and

(ii) the FI,

$$\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = \sum_{k=1}^n \{g_1, \dots, \{f_1, \dots, f_{n-1}, g_k\}, \dots, g_n\}. \tag{8}$$

The commutative algebra  $\mathcal{A}$ , equipped with a Nambu–Loday bracket, will be called Nambu–Loday algebra. For Nambu–Loday algebras we have no direct inductive characterization like that for Nambu–Poisson and Nambu–Jacobi brackets [GM1], since the property saying that fixing an argument we obtain a bracket satisfying FI, but of one argument less, is based on skew symmetry. However, we can prove the following.

**Theorem 1.** *If  $\mathcal{A}$  is an associative commutative algebra over a field of characteristic zero and  $\mathcal{A}$  contains no nilpotents, then every Nambu–Loday bracket on  $\mathcal{A}$  is skew symmetric.*

**Proof.** Let us assume that we have fixed a Nambu–Loday bracket on  $\mathcal{A}$ . First, observe that the skew-symmetry property is equivalent to the fact that the bracket vanishes, if only two arguments are the same. Explicitly, if

$$\{f_1, \dots, f_n\} = 0 \text{ for all } f_1, \dots, f_n \in \mathcal{A} \text{ with } f_i = f_j = h \text{ for some } i \neq j \tag{9}$$

then, writing  $h = x + y$  and using (9) for  $h = x$  and  $y$ , we obtain

$$\begin{aligned} &\{f_i, \dots, f_{i-1}, x, f_{i+1}, \dots, f_{j-1}, y, f_{j+1}, \dots, f_n\} \\ &= -\{f_i, \dots, f_{i-1}, y, f_{i+1}, \dots, f_{j-1}, x, f_{j+1}, \dots, f_n\}. \end{aligned} \tag{10}$$

Second, since we can obtain the skew symmetry (10) with respect to the transposition  $(i, j)$  composing transpositions  $(i, n)$ ,  $(j, n)$  and  $(i, n)$  again, it is sufficient to prove (10) (or (9)) for  $j = n$ .

Fix  $i = 1, \dots, n - 1$ . Replacing  $f_i$  in (8) by  $f_i^2/2$ , we obtain, due to the Leibniz rule,

$$f_i \{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = \sum_{k=1}^n \{g_1, \dots, f_i \{f_1, \dots, f_{n-1}, g_k\}, \dots, g_n\}. \tag{11}$$

Subtracting from (11) the FI (8) multiplied by  $f_i$ , we obtain

$$\sum_{k=1}^n (\{f_1, \dots, f_{n-1}, g_k\} \{g_1, \dots, g_{k-1}, f_i, g_{k+1}, \dots, g_n\}) = 0 \tag{12}$$

which holds for all  $i = 1, \dots, n - 1$ .

Now, for  $n - 1 \geq m \geq 1$ , we shall show inductively the following:

$$(S_m) \text{ If } m \text{ elements of } f_1, \dots, f_{n-1} \in \mathcal{A} \text{ equal } h \text{ then } \{f_1, \dots, f_{n-1}, h\} = 0. \tag{13}$$

Of course,  $(S_1)$  tells us just that the bracket is skew symmetric with respect to all transpositions  $(i, n)$ , so it is totally skew symmetric, according to the previous remarks.

We start with  $m = n - 1$ . Putting in (12) all  $f$  and  $g$  equal to  $h$ , we obtain  $n\{h, \dots, h\}^2 = 0$ , which gives us  $\{h, \dots, h\} = 0$ , since there are no nilpotents in  $\mathcal{A}$ , so the induction starts. To prove the inductive step, assume  $(S_m)$  for some  $n - 1 \geq m > 1$ . We shall show  $(S_{m-1})$ . Take  $f_1, \dots, f_n \in \mathcal{A}$  such that  $f_j = h$  for  $j$  from a subset  $I$  of  $\{1, \dots, n - 1\}$  with  $(m - 1)$  elements. Put  $f_k = g_k, k = 1, \dots, n - 1$  and  $g_n = h$ . For a fixed  $i = 1, \dots, n - 1$ , we have:

- (i) if  $k \notin I$ , then  $\{g_1, \dots, g_{k-1}, f_i, g_{k+1}, \dots, g_n\} = 0$  by the inductive assumption, and
- (ii) if  $k \in I$ , then  $\{f_1, \dots, f_{n-1}, g_k\} \{g_1, \dots, g_{k-1}, f_i, g_{k+1}, \dots, g_n\} = \{f_1, \dots, f_{n-1}, h\}^2$ .

This implies that (12) reads in this case

$$m\{f_1, \dots, f_{n-1}, h\}^2 = 0 \quad (14)$$

which gives

$$\{f_1, \dots, f_{n-1}, h\} = 0 \quad (15)$$

for any  $f_1, \dots, f_{n-1} \in \mathcal{A}$  such that  $(m-1)$  of them equal  $h$ .  $\square$

**Remark.** The assumption that there are no nilpotents in  $\mathcal{A}$  is essential. To see this, consider the commutative associative algebra  $\mathcal{A}$  over  $k$  freely generated by  $x, y$ , with the constraint  $x^2 = 0$ , i.e.  $\mathcal{A} = k[x, y]/\langle x^2 \rangle$ . It is easy to see that the bracket

$$\{f, g\} = x \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} \quad (16)$$

satisfies the Leibniz rule and the Jacobi identity, but it is symmetric.

### 3. Generalized Lie algebroids

Every  $n$ -ary bracket on the algebra  $C^\infty(M)$  of smooth functions on a manifold  $M$ , which satisfies the Leibniz rule, is associated with an  $n$ -contravariant tensor  $\Lambda$  according to

$$\{f_1, \dots, f_n\} = \langle \Lambda, df_1 \wedge \dots \wedge df_n \rangle. \quad (17)$$

The vector fields

$$\Lambda_{(f_1, \dots, f_{n-1})} = \{f_1, \dots, f_{n-1}, \cdot\} = i_{df_1 \wedge \dots \wedge df_{n-1}} \Lambda \quad (18)$$

we can call (left) *Hamiltonian vector fields* of  $\Lambda$ . It is easy to see that the FI for the  $n$ -bracket is in this case equivalent to the fact that the Hamiltonian vector fields preserve the tensor  $\Lambda$ , i.e.

$$\mathcal{L}_{\Lambda_{(f_1, \dots, f_{n-1})}} \Lambda = 0 \quad (19)$$

where  $\mathcal{L}$  stands for the Lie derivative. Theorem 1 can be formulated in this case as follows.

**Corollary 1.** *If an  $n$ -contravariant tensor field is preserved by its Hamiltonian vector fields, then it is skew symmetric.*

It is well known that with an  $n$ -ary bracket on a finite-dimensional vector space  $V$  (over  $\mathbb{R}$ ) we canonically associate a linear contravariant  $n$ -tensor  $\Lambda$  on the dual space  $V^*$  such that (17) is satisfied for linear functions on  $V^*$  (thus elements of  $V$ ). Explicitly, if  $(x_1, \dots, x_k)$  is a basis of  $V$  (thus a coordinate system of  $V^*$ ), then

$$\Lambda = \sum_{i_1, \dots, i_n=1}^k \{x_{i_1}, \dots, x_{i_n}\} \partial_{i_1} \otimes \dots \otimes \partial_{i_n}. \quad (20)$$

Lie algebras correspond in this way to linear Poisson tensors. This can be generalized to vector bundles as follows. By linear functions on a vector bundle  $E$  over a manifold  $M$  we understand the functions we obtain from sections of the dual bundle  $E^*$  by contraction, i.e. the linear function  $\iota_X$  associated with a section  $X$  of  $E^*$  is given by  $\iota_X(\alpha(p)) = \langle X(p), \alpha(p) \rangle$ , where  $\alpha(p) \in E_p$  for  $p \in M$ . We say that an  $n$ -tensor  $\Lambda$  on  $E^*$  is linear if linear functions on  $E^*$  are closed with respect to the  $n$ -ary bracket  $\{\cdot, \dots, \cdot\}_\Lambda$  generated by  $\Lambda$ . Hence, we can define an  $n$ -ary operation  $[\cdot, \dots, \cdot]_\Lambda$  on sections of the bundle  $E$  by

$$\iota_{[X_1, \dots, X_n]_\Lambda} = \{\iota_{X_1}, \dots, \iota_{X_n}\}_\Lambda. \quad (21)$$

In [GU1, GU2] this idea was used to define general (binary) algebroid structures, and in [GM2] to define  $n$ -ary Lie algebroids. Let us concentrate now on the binary case and let us recall from [GU2] the following definition.

**Definition.** *Let  $M$  be a manifold. An algebroid on  $M$  is a vector bundle  $\tau : E \rightarrow M$ , together with a bracket  $[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  on the module  $\mathcal{A} = \Gamma E$  of global sections of  $E$ , and two vector bundle morphism  $a_l, a_r : E \rightarrow TM$ , over the identity on  $M$ , from  $E$  to the tangent bundle  $TM$ , called the anchors of the Lie algebroid (left and right), such that*

$$[fX, gY] = fg[X, Y] + fa_l(X)(g)Y - ga_r(Y)(f)X \tag{22}$$

for all  $X, Y \in \mathcal{A}$  and all  $f, g \in \mathbb{C}(M)$ .

It is clear that any finite-dimensional algebra structure can be viewed as an algebroid structure on a bundle over a single point. Note that in the case when the algebroid bracket is a Lie bracket, we have  $a_l = a_r = a$  and  $a([X, Y]) = [a(X), a(Y)]$  for all  $X, Y \in \mathcal{A}$ . Such structures are called *Lie algebroids*. They were introduced by Pradines [Pr] as infinitesimal objects for differentiable groupoids, but one can find similar notions proposed by several authors in increasing number of papers (which proves their importance and naturalness). For basic properties and the literature on the subject we refer to the survey article by Mackenzie [Ma].

**Theorem 2 ([GU1]).** *There is a one-to-one correspondence between linear two-contravariant tensors  $\Lambda$  on the dual bundle  $E^*$  and algebroid brackets  $[\cdot, \cdot]_\Lambda$  on  $E$ .*

Note that, equivalently, we can think of algebroid structures on the vector bundle  $E$  as morphisms of double vector bundles  $\varepsilon : T^*E \rightarrow TE^*$  (cf [GU2]).

We can speak about *Loday algebroids* when we impose the Jacobi identity (3) but we skip the skew-symmetry assumption. One can think that, imposing the Jacobi identity for an algebroid, we obtain the Jacobi identity for the bracket  $\{\cdot, \cdot\}_\Lambda$  of functions defined by the corresponding tensor  $\Lambda$  on  $C^\infty(E^*)$  and, in view of theorem 1, that this implies that  $\Lambda$  is a Poisson tensor, so our algebroid is a Lie algebroid. This reasoning, however, is wrong, since the Jacobi identity on sections of  $E$  forces the Jacobi identity for the bracket  $\{\cdot, \cdot\}_\Lambda$  only for linear functions. Such tensors may be non-skew-symmetric; i.e., clearly, Loday algebras do exist. A simple example is the following.

**Example 1.** *Consider the two-tensor on  $\mathbb{R}^3$  given by*

$$\Lambda = x_2\partial_1 \otimes \partial_1 + x_3\partial_1 \otimes \partial_3 - x_3\partial_3 \otimes \partial_1. \tag{23}$$

*It is easy to see that the Hamiltonian vector fields of linear functions preserve  $\Lambda$ , so we have the Jacobi identity for the associated bracket:*

$$[x_1, x_1] = x_2 \quad [x_1, x_3] = -[x_3, x_1] = x_3 \tag{24}$$

*where we assume the missing brackets to be zero. This example is also an example of a Loday algebroid over a single point, but we can obtain a Loday algebroid over  $M$  just by tensoring the above algebra with  $C^\infty(M)$ .*

The anchors of the Loday algebroids from the above example are trivial. We shall show that this is not incidental and Loday algebroids can be reduced to Lie algebroids and bundles of Loday algebras.

**Theorem 3.** *For any Loday algebroid bracket the left anchor is the same as the right anchor and the bracket is skew symmetric at points where they do not vanish.*

**Proof.** Let  $[\cdot, \cdot]$  be a Loday algebroid bracket on the space  $\mathcal{A}$  of sections of a vector bundle  $E$  over  $M$ . The Jacobi identity implies immediately

$$[[X, X], Y] = 0 \quad (25)$$

for all  $X, Y \in \mathcal{A}$ . Putting  $X := fX$  in (25), we obtain

$$f(a_l(X)(f) - a_r(X)(f))[X, Y] - 2fa_r(Y)(f)[X, X] - fa_r(Y)(a_l(X)(f) - a_r(X)(f))X - (a_l(X)(f) - a_r(X)(f))a_r(Y)(f)X = 0 \quad (26)$$

for all  $X, Y \in \mathcal{A}$  and all  $f \in C^\infty(M)$ . Suppose that at  $p \in M$  the right anchor does not vanish; i.e., there are  $Y \in \mathcal{A}$  and  $f \in C^\infty(M)$  such that  $a_r(Y)(f)(p) \neq 0$ . We can additionally assume that  $f(p) = 0$  and then (26) implies that  $(a_l(X)(f) - a_r(X)(f))(p) = 0$  for all  $X \in \mathcal{A}$ . Hence the vector  $(a_l(X) - a_r(X))(p)$  annihilates any covector from  $T_p^*M$  not annihilated by  $a_r(Y)(p)$ , thus it is zero. However, if  $a_r$  does not vanish at  $p$ , then it does not vanish in a neighbourhood of  $p$ , so  $a_l(X) = a_r(X)$  in a neighbourhood of  $p$  and (26) implies now that in this neighbourhood

$$fa_r(Y)(f)[X, X] = 0 \quad (27)$$

for all  $X, Y \in \mathcal{A}$  and  $f \in C^\infty(M)$ . Since  $a_r$  is nontrivial in this neighbourhood, this in turn implies  $[X, X] = 0$ ; i.e., the bracket is skew symmetric. In particular, the left anchor is equal to the right one.

Assume now that the right anchor vanishes at  $p \in M$ . By (26) we obtain now

$$f(p)a_l(X)(f)(p)[X, Y](p) = 0 \quad (28)$$

for all  $X, Y \in \mathcal{A}$  and  $f \in C^\infty(M)$ , so

$$a_l(X)(f)(p)[X, Y](p) = 0. \quad (29)$$

Replacing  $X$  in (29) by  $X + Z$ , we obtain

$$a_l(X)(f)(p)[Z, Y](p) + a_l(Z)(f)[X, Y](p) = 0. \quad (30)$$

Multiplying the above equation by  $a_l(X)(f)(p)$  and taking into account (29), we obtain

$$(a_l(X)(f)(p))^2[Z, Y](p) = 0 \quad (31)$$

for all  $X, Y, Z \in \mathcal{A}$  and  $f \in C^\infty(M)$ , which clearly implies that the left anchor vanishes at  $p$ , since, if the bracket is trivial at  $p$ , then both anchors are trivial at  $p$ . Hence, the right anchor is the same as the left anchor and the bracket is skew symmetric at points where they do not vanish.  $\square$

#### 4. Conclusions

Poisson and Lie algebroid brackets are some of the most fundamental algebraic structures in classical and quantum physics. We have composed the two ways of generalizing the Poisson bracket: the Nambu idea of an  $n$ -ary bracket and the Loday observation that skipping the skew-symmetry assumption in the definition of a Lie algebra we still have a (co-) homology theory. What we obtain is that no new structures appear in this way, since the Leibniz rule and the FI imply the skew symmetry. A similar phenomena is found when looking for a non-skew version of a Lie algebroid. This shows that skew symmetry is in fact forced by other properties of these important algebraic structures.

It would be interesting to know whether the same is true for more general brackets, such as Nambu–Jacobi brackets or brackets acting as multidifferential operators. If we skip skew symmetry, then it is not even clear whether the latter have to be of first order. We can prove the skew symmetry for binary Nambu–Jacobi–Loday brackets and hope that the methods used in

the proof of an algebraic version of the well known Kirillov theorem on local Lie algebras ([Gr], theorem 4.2) can be of some help in proving a general result. We postpone these studies to a separate paper.

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